Diffusion in free turbulent shear flows

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SUMMARY

This paper is concerned with some statistical properties of the displacement of a marked fluid particle released from a given position in a turbulent shear flow, and in particular with the dispersion about the mean position after a long time. It is known that the dispersion takes a simple asymptotic form when the particle velocity is a stationary random function of time, and that analogous results are obtainable when the particle velocity can be transformed to a stationary random function by suitable stretching of the velocity and time scales. The basic hypothesis of the paper is that, in steady free turbulent shear flows which are generated at a point and have a similar structure at different stations downstream, the velocity of a fluid particle exhibits a corresponding Lagrangian similarity and can be so transformed to a stationary random function.

The velocity and time scales characterizing the motion of a fluid particle at time t after release at the origin are determined in terms of the powers with which the Eulerian length and velocity scales of the turbulence vary with distance x from the origin. The time scale has the same dependence on t for all jets, wakes and mixing layers (and also for decaying homogeneous turbulence) possessing the usual kind of Eulerian similarity. The dispersion of a particle in the longitudinal or mean-flow direction (and likewise that in the lateral direction in cases of two-dimensional mean flow) is found to vary with t in such a way as to be proportional to the thickness of the shear layer at the mean position of the particle. The way in which the maximum value of the mean concentration of marked fluid falls off with t (for release of a single particle) or with x (for continuous release) is also found.

1. INTRODUCTION

The purpose of this paper is to show that the well-known analysis of the dispersion of a marked fluid particle due to turbulence can be adapted to a discussion of diffusion in (statistically) steady turbulent jets, wakes and mixing layers. These cases, which hitherto have lain outside the scope of the available analysis, are distinguished by their property that the mean velocity is only approximately unidirectional. As a preliminary to an explanation of the modification required for these new cases, the existing analysis will be described briefly.

The object, in either a theoretical or an experimental investigation of turbulent diffusion of some conserved quantity not subject to molecular diffusion, is essentially to determine the statistical history of a volume of marked fluid which occupies a given position at an initial instant. If only the probability of finding marked fluid at any point and at any subsequent time (which is effectively the local mean concentration of the quantity undergoing diffusion) is required, it is sufficient to consider each volume element of the marked fluid in isolation. Only this limited aspect of turbulent diffusion will be considered herein. We are thus concerned with the statistical history of a single small element of fluid whose position is given at an initial instant. A description of the place of such 'one-particle analysis' in the general framework of the theory of turbulent diffusion will be found in a recent survey (Batchelor & Townsend 1956).

It will be enough for us to consider one component of the vector position of the fluid particle or volume element under discussion. Thus, we wish to determine the probability distribution of the displacement $X(t)$ which the particle has undergone during the time interval t subsequent to the initial instant at which its position was given. The displacement $X(t)$ is a random quantity, taking different values for each trial or realization of the turbulent flow, and averages will be regarded as referring to an ensemble of such realizations. From a practical point of view, the most useful parameters of the probability distribution of $X(t)$ are the mean displacement

$$\overline{X}(t) = \int_0^t \overline{u}(t')\,dt',$$

where $u(t)$ is (one component of) the particle velocity at time t, and the dispersion $D_x(t)$ about the mean displacement, where

$$[D_x(t)]^2 = \overline{[X(t)-\overline{X}(t)]^2}.$$

G. I. Taylor (1921) pointed out that there were advantages in thinking of the dispersion in terms of the particle velocity by means of the relation

$$\begin{aligned}\frac{dD_x^2}{dt} &= 2\overline{[u(t)-\overline{u}(t)][X(t)-\overline{X}(t)]}\\ &= 2\int_0^t \overline{[u(t)-\overline{u}(t)][u(t')-\overline{u}(t')]}\,dt'.\end{aligned}$$

Taylor also noted—and this is the vital point in this well-known analysis—that it was possible to obtain some simple and definite results in the case in which $u(t)$ is a stationary random function of t, i.e. when the statistical properties of the particle velocity are constant in time. We then have

$$\overline{X}(t) = t\overline{u},$$

$$\overline{[u(t)-\overline{u}][u(t')-\overline{u}]} = R(t-t'),$$

and

$$\frac{dD_x^2}{dt} = 2\int_0^t R(\xi)\,d\xi,$$

R being the covariance function of the fluctuation in particle velocity. It will normally happen, in cases of well defined turbulent flow, that $R(\xi) \rightarrow 0$

as $\xi \to \infty$ and that at least the first few integral moments of $R(\xi)$ converge. Consequently we have the important result that, as $t \to \infty$,

$$D_x^2 \to 2At - 2B,$$

where (as found from integration by parts),

$$A = \int_0^\infty R(\xi)\, d\xi, \qquad B = \int_0^\infty \xi R(\xi)\, d\xi.$$

Moreover, since the displacement can be written in the form

$$X(t) = \sum_{n=1}^{t/\alpha} \int_{(n-1)\alpha}^{n\alpha} u(t')dt',$$

which is a series of random terms of equal variance, it follows from a plausible application (although a non-rigorous one, because the terms of the series are not statistically independent) of the central limit theorem that, as $t \to \infty$ (with α remaining constant), the probability distribution of $X(t)$ tends to the normal or Gaussian form (Batchelor 1949). When $X(t)$ is normally distributed, the probability of finding the fluid particle within a given small range of positions at time t satisfies a Fickian or heat conduction type of equation and the above integral denoted by A can then be interpreted as a diffusivity, or coefficient of diffusion, with dimensions (velocity) × (length).

In short, it is possible to obtain some useful results, and in particular to determine the asymptotic dependence of the dispersion on the time, when the random velocity of the particle is a stationary function of time. We are thus led to look about for turbulent flows in which the particle velocity has this simple property—or, failing stationarity, some related property which might allow a modified form of the above analysis to be applied.

2. Two previous applications of one-particle analysis

The first case of turbulent motion in which a study of particle dispersion was made was decaying turbulence generated by a grid placed in a uniform stream (Taylor 1935), and this remained the only one for many years. As a consequence of the decay of the turbulence, the velocity of a fluid particle is not here a stationary random function of time (not even approximately), and some modification of the above analysis is necessary before it can be applied. Provided it may be assumed that the turbulence preserves its structure as it decays (so that all functions characterizing the turbulence have the same form, although possibly with different length, time and velocity scales, at all relevant stages of the decay), a suitable modification of the diffusion analysis can be made to allow for the decay (Batchelor 1952; Batchelor & Townsend 1956). The essential point of the modification is to make a mathematical transformation of the particle velocity—expanding the velocity scale and contracting the time scale more and more as the decay proceeds—such that it *becomes* a stationary random function of a new variable related to time in a known way. (Taylor (1935) attempted to allow for the decay of the turbulence, but his method is not wholly correct inasmuch as it takes into account only the change in the velocity scale and ignores the possibility of an independent change in the time scale of the motion.) The modification that will be adopted to make the diffusion analysis applicable to free turbulent shear flows is similar in principle to that used for decaying turbulence behind a grid.

A few years ago, when Sir Geoffrey Taylor was conducting his experiments on the longitudinal extension of a finite volume of salt solution injected into the turbulent flow of water along a circular pipe (Taylor 1954), I realized that turbulent flow along a cylindrical pipe is a case to which the above one-particle analysis can be applied without the need for any modification. The two properties of the turbulent flow in a pipe that together ensure that the velocity of a particle is a stationary random function of time are: (*a*) the fluid particle is constrained by the walls always to lie within the turbulent flow inside the pipe; (*b*) the turbulence has the same statistical properties at all cross-sections of the pipe. Even though the fluid particle may move into different parts of the pipe cross-section, including the slow moving layer near the wall, the statistical properties of the particle velocity do not change as it moves along the pipe, and the above analysis of diffusion (with x and u referring to the longitudinal or axial direction) is immediately applicable. Some observations of the times of travel of small solid spheres between two distant cross-sections of a circular pipe have been analysed with the aid of this diffusion analysis (Batchelor, Binnie & Phillips 1955).

We are now in a position to consider the modification necessary to render the diffusion analysis applicable to the class of free turbulent shear flows, of which jets, wakes, and mixing layers are examples. Free turbulent shear flows differ from turbulent flow in a pipe, firstly in that the turbulence is not bounded by a rigid wall but adjoins non-turbulent fluid of uniform mean velocity, and secondly, in that, as a consequence of the absence of the rigid cylindrical wall, the turbulence changes from one cross-section to another, the simplest manifestation of this change being an increase in the width of the turbulent flow with increase of distance along the streamlines of the mean flow. So far as this second difference is concerned, it will often happen, in cases of free turbulent shear flow, that the turbulence preserves its structure as it changes with distance downstream, and that the change in the turbulence is confined to changes in the length, time and velocity scales of the motion; such a state of self-preservation is usually set up at a sufficient distance from the source or origin of the turbulence*.

* Turbulent boundary layers on rigid walls have one ‘ free ’ boundary and thereby might be thought to qualify for inclusion within the group of shear flows under discussion. However, boundary layers on uniformly rough walls are different from the other examples named inasmuch as a self-preserving state of the turbulence is attained here only in a certain limited approximate sense, even when the Reynolds number is large. (In the rather artificial case of a uniform stream passing over a rigid wall with linearly increasing roughness height, exact self-preservation of the boundary layer turbulence is possible; the diffusion laws are then the same as those to be described later for the case of a mixing layer with zero velocity in the stream on one side.) In what follows, only those free turbulent shear flows which attain a self-preserving state exactly (possible only asymptotically, i.e. far downstream) will be considered.

We have here a situation which is a little like that in the case of diffusion in decaying turbulence behind a grid, and a natural suggestion is to employ the same trick of transforming the velocity of a fluid particle in such a way that it becomes a stationary random function. However, before exploring this suggestion, there is a question arising out of the first of the above differences between free turbulent shear flow and confined turbulent shear flow which must be answered.

3. Can a fluid particle escape from the region of turbulent motion?

As remarked above, the presence of the walls of a pipe ensures that a fluid particle remains within the region of turbulent flow. Is there a similar guarantee, in the case of free turbulent shear flows, that a particle remains always in the region of turbulent flow? Or is it possible that some fluid elements are ejected from the region of turbulent flow into the surrounding non-turbulent fluid and subsequently take no part in the turbulent motion? It is necessary to settle this question before proceeding to an analysis of turbulent diffusion, because, if the possibility of such an escape from a region of turbulent flow to a region of non-turbulent flow should exist, it would have a profound effect on the statistical properties of the particle velocity. The velocity of the fluid particle would fall rapidly to zero, and might stay at that value since a return to the region of turbulent flow would not be inevitable. (This is in contrast to the case of flow in a pipe; the viscous sub-layer near the walls may perhaps be regarded here as a region of non-turbulent flow in the particular sense that inertia forces are not important, but the velocity of a particle remains random in the viscous sub-layer and, in view of the geometrical contraint of the pipe wall, there is statistical certainty that a particle in the viscous sub-layer will eventually return to the central region of the pipe.) In the event of the probability of escape from the region of turbulent flow being finite, it would be impossible to transform the particle velocity into a stationary random function by simple adjustment of the velocity and time scales.

Fortunately, the available evidence about the nature of free turbulent shear flows suggests that such escape does not occur (see Corrsin & Kistler 1954). It has been known for some years that free turbulent shear flows are characterized, instantaneously, by a sharp boundary, of irregular and random form, separating a central region of turbulent motion from an outer region in which the flow is non-turbulent in the sense that the vorticity is zero. Something of the mechanism by which this instantaneous boundary remains sharp against the smearing action of viscosity is known. It seems that vorticity is diffused, by the action of viscosity, from the central to the outer region, and that the high rate of stretching of vortex lines in the central region rapidly increases the magnitude of the vorticity to some high equilibrium level as soon it is made finite by viscous diffusion; in this way the sharp boundary propagates relative to the non-turbulent fluid in the outer region. What is important in the present connection is that the boundary always advances into the non-turbulent fluid, thus acting as a valve which allows non-turbulent fluid to pass into the central region by mixing or entrainment and to be converted to turbulent fluid, but which does not allow fluid to pass out of the central region.

It seems, therefore, that, if a fluid element is once inside the central region of turbulent motion, it remains within it. The general way in which the velocity of a fluid particle in the central region changes with time will thus be related to the fact that, as the particle moves downstream (always remaining within the central region), it is subject to the influence of turbulent motion whose length and time scales are continually changing. We now consider how the velocity of a fluid particle in the central region of turbulent motion may be transformed into a stationary random function.

4. Transformation of the particle velocity for flows with similarity

We shall suppose that each of the turbulent shear flows concerned is steady and has the same structure at different distances x downstream from some virtual origin. The only change in the statistical properties of the flow (including the variation of the Eulerian mean velocity, although not necessarily its absolute magnitude) at different stations downstream is a change of the length (L) and velocity (V) scales of the motion. It will also be assumed that these length and velocity scales are proportional to powers of x, i.e.

$$L(x) \propto x^{p}, \qquad V(x) \propto x^{-q}, \tag{4.1}$$

the corresponding variation of the time scale of the motion being as x^{p+q}. Both these assumptions are usually valid at sufficiently large Reynolds numbers and at sufficiently large distances from the real origin for the common cases of free turbulent shear flow which are not influenced by rigid boundaries.

An element of fluid is carried downstream, and the statistical properties of its velocity $u(t)$ change with t as a consequence of the variation of the properties of the Eulerian velocity field with position x. Whatever may be the distance of the element downstream, it finds itself surrounded always by turbulence of the same statistical form. Thus the fluctuations of the velocity of the fluid particle have a (statistically) similar form at different times. We are therefore led to make the hypothesis that the particle velocity can be transformed to a stationary random function by suitable adjustment of the velocity and time scales. (More precisely, it is not $u(t)$, but $u(t)-U_0$, which is so transformable, where U_0 is a velocity of translation of the whole field which has no effect on the turbulence and serves only to give a frame of reference with respect to which the flow is statistically steady; U_0 is finite in the case of a wake, and is usually zero otherwise.) Analytically, the hypothesis is that

$$[u(t)-U_0]/\omega(t) = F(\eta), \tag{4.2}$$

where $F(\eta)$ is a stationary random function of a new variable η given by

$$d\eta \propto dt/\tau(t), \tag{4.3}$$

and $\omega(t)$ and $\tau(t)$ are the velocity and time scales of the statistical properties of the particle's motion at time t. The origin of t can be taken as the instant of release of the particle from a fixed position, and it will be necessary for validity of the hypothesis that t be large enough for the exact circumstances of the release to be 'forgotten'. The hypothesis represented by (4.2) and (4.3) is no more than a similarity hypothesis for the Lagrangian features of the turbulence, exactly analogous to the better known similarity hypothesis for the Eulerian features. It is probable that the one similarity hypothesis is a strict consequence of the other, but attempts to prove this encounter the usual difficulties of relating Lagrangian and Eulerian features of flow.

We have now to determine $\omega(t)$ and $\tau(t)$, making use of the information that the velocity and time scales of the Eulerian properties of the turbulence at distance x downstream are proportional to x^{-q} and x^{p+q} respectively. Now the mean distance downstream of the particle (which will be supposed, for the sake of simplicity of algebraic form of the relations that follow, to have been released at $x = 0$, i.e. at the origin of the turbulence) at time t is $\overline{X}(t)$, and the velocity and time scales of the Eulerian features of the turbulence at the position $x = \overline{X}(t)$ will presumably be also the velocity and time scales of the statistical properties of the particle motion at time t. Thus we have

$$\omega(t) \propto [\overline{X}(t)]^{-q}, \qquad \tau(t) \propto [\overline{X}(t)]^{p+q}. \tag{4.4}$$

The mean velocity of the particle is itself one of the statistical quantities included in the similarity hypothesis, and the dependence of $\overline{X}(t)$ on t can be determined from

$$\frac{d\overline{X}(t)}{dt} = \overline{u}(t) = U_0 + \overline{F}(\eta)\omega(t), \tag{4.5}$$

in which $\overline{F}(\eta)$ is a constant.

Equations (4.2) and (4.3), together with the auxiliary relations (4.4) and (4.5), define the transformation that enables a modified form of the diffusion analysis described earlier to be applied to free turbulent flows whose properties change with distance downstream.

The form of $\overline{X}(t)$, as determined by integration of (4.5), depends on whether U_0 is zero or not. It is convenient therefore to give separate consideration to these two cases in the next two sections.

5. Diffusion in jet-type flows

We include under this heading all those steady free turbulent shear flows in which, as in the typical case of a jet discharging into stationary fluid, the *absolute* value of the velocity at any point conforms to the similarity laws represented by (4.1), so that $U_0 = 0$. Integration of (4.5), with the aid of (4.4), then gives

$$\overline{X}(t) \propto t^{1/(1+q)}, \tag{5.1}$$

the constant of integration being determined by the condition that the fluid particle is released at the position $x = 0$.

The new independent variable η is thus defined by

$$d\eta = t^{-(p+q)/(1+q)}\,dt,$$

the arbitrary multiplicative constant being chosen as unity for convenience. Integration leads to two different forms of relation between η and t according as $p = 1$ or $p \neq 1$. It is not necessary to consider both cases, because, for all the types of flow concerned in this section, similarity of structure of the flow at different distances downstream is possible only with $p = 1$. This may be seen from a comparison of the two terms $U\,\partial U/\partial x$ and $\partial \overline{uv}/\partial y$ which occur in the Eulerian equations of mean motion (the symbols having their usual meanings and not those applicable elsewhere in this paper). When both U^2 and $\overline{uv}$ have the form required for (Eulerian) similarity, viz.

$$x^{-2q} \times \text{function of } y/x^p,$$

the equation can be satisfied only if $p = 1$. Jets and mixing layers (with zero velocity on one side) are thus straight-sided, the turbulence being contained in either a conical or a wedge-shaped region.

The relation between η and t is thus

$$\eta = \log t. \tag{5.2}$$

The fluctuation of the particle velocity about its mean value is

$$u(t) - \overline{u}(t) \propto t^{-q/(1+q)}[F(\eta)\overline{F} - (\eta)]$$
$$= t^{-q/(1+q)}f(\eta), \quad \text{say},$$

where $f(\eta)$ is likewise a stationary random function of η, and the fluctuation of the particle position about its mean value is

$$X(t) - \overline{X}(t) = \int_0^t t'^{-q/(1+q)} f(\eta')\,dt'$$
$$= \int_{-\infty}^{\log t} e^{\eta'/(1+q)} f(\eta')\,d\eta'. \tag{5.3}$$

We note in passing that the presence of the weighting factor in this integral will lead to the value of $X(t) - \overline{X}(t)$ being dominated by a finite portion of the range of integration, even when $t \to \infty$; in these circumstances, it would not follow from the central limit theorem that the probability distribution of $X(t) - \overline{X}(t)$ tends to the Gaussian form as $t \to \infty$.

The relation from which the statistical dispersion of the particle is determined now becomes

$$\frac{dD_x^2}{dt} = t^{-q/(1+q)} \int_{-\infty}^{\log t} e^{\eta'/(1+q)}\overline{f(\eta)f(\eta')}\,d\eta'$$
$$= t^{(1-q)/(1+q)} \int_0^{\infty} e^{-\zeta/(1+q)} R(\zeta)\,d\zeta, \tag{5.4}$$

where $\zeta = \eta - \eta'$ and $R(\zeta)$ is the covariance function of $f(\eta)$. If the position of release of the particle had been taken as some finite value of x, the upper terminal of this integral would have been finite; however, the integral is convergent and the asymptotic form of the dispersion (as $t \to \infty$) would still be given by the above relation. Integration of (5.4) gives

$$D_x(t) \propto t^{1/(1+q)} \propto \overline{X}(t). \tag{5.5}$$

The striking feature of this result is that the longitudinal dispersion of the marked fluid particle increases in proportion to the thickness of the shear layer at the mean position of the particle. Comment on this feature is postponed until § 8.

If it happens that the probability distribution of the displacement of the fluid particle has a Gaussian form (which may be so in some cases, although there are as yet insufficient grounds for expecting the Gaussian form in general), the quantity $\frac{1}{2}dD_x^2/dt$ can be interpreted as a diffusivity. We might then account for the power of t in the expression for $\frac{1}{2}dD_x^2/dt$ by noting that $t^{1/(1+q)}$ is a measure of the mean distance downstream of the fluid particle at time t, and that representative length and velocity scales of the turbulent diffusing motions at distance x downstream are proportional to x and x^{-q} respectively.

6. Diffusion in wake-type flows

In the case of a wake behind a body held fixed in an otherwise uniform stream of speed U_0, only the velocities in the shear flow relative to U_0 conform to the self-preservation laws, and the first term on the right hand side of (4.5) is non-zero. Moreover, a turbulent wake has a self-preserving structure only when the variations of velocity across the wake are small compared with U_0, so that we might as well neglect the second term on the right hand side of this equation. With this approximation that the mean position of the particle is the same as if it travelled with the speed of the undisturbed stream, i.e. that

$$\overline{X}(t) = U_0 t, \tag{6.1}$$

we have

$$d\eta = t^{-p-q}\,dt$$

(the multiplicative constant again being put equal to unity). Again integration gives two different forms of the relation between η and t according as $p+q = 1$ or $p+q \neq 1$, and again an additional condition for self-preservation of the turbulence allows only one of these possibilities. As before, we compare the ways in which the terms $U\,\partial U/\partial x$ and $\partial\overline{uv}/\partial y$ in the Eulerian equation of motion depend on x; on noting that the appropriate approximate form of $U\,\partial U/\partial x$ is $U_0\,\partial U/\partial x$ in the present case, we find

$$p+q = 1.$$

Thus the relation between η and t is again

$$\eta = \log t.$$

The fluctuation of the particle velocity about its mean value is now

$$\begin{aligned} u(t) - \bar{u}(t) &\propto (U_0 t)^{-q}[F(\eta) - \bar{F}(\eta)] \\ &= t^{-q} f(\eta), \quad \text{say,} \end{aligned}$$

and the fluctuation of the particle position about its mean value is

$$X(t) - \overline{X}(t) = \int_{-\infty}^{\log t} e^{(1-q)\eta'} f(\eta')\,d\eta'. \tag{6.2}$$

The relation from which the dispersion can be determined is

$$\frac{dD_x^2}{dt} = t^{-q}\int_{-\infty}^{\log t} e^{(1-q)\eta'}\overline{f(\eta)f(\eta')}\,d\eta'$$

$$= t^{1-2q}\int_0^{\infty} e^{-(1-q)\zeta}\,R(\zeta)\,d\zeta, \tag{6.3}$$

showing that

$$D_x(t) \propto t^{1-q} \propto [\overline{X}(t)]^p. \tag{6.4}$$

The general remarks in the preceding section are also relevant here.

7. Lateral diffusion in two-dimensional free turbulent shear flows

The notation and wording of the preceding sections have been chosen to refer to diffusion in the longitudinal direction, i.e. in the direction of the mean flow. However it is evident that, in cases in which the shear flow is statistically two-dimensional, the above analysis can also be regarded as describing diffusion in the lateral y-direction (that in which the turbulence is stationary). The convection of the marked particle in the longitudinal direction by the mean flow again determines the way in which the properties of the turbulence in the neighbourhood of the particle change with time, and lateral diffusion follows laws of the same form as for longitudinal diffusion, the only difference being that there is no mean displacement of the particle in the lateral direction. Thus, the lateral component of the velocity of the particle is of the form

$$v(t) = \begin{cases} t^{-q/(1+q)}g(\eta) & \text{for jet-type flows,} \\ t^{-q}g(\eta) & \text{for wake-type flows,} \end{cases}$$

where g is a stationary random function of η (which is related to t as in the two preceding sections) with zero mean. Then, if $Y(t)$ is the lateral displacement of the particle from its initial position in time t, the mean value of $Y(t)$ is zero, and the dispersion about the mean is given asymptotically, as $t \to \infty$, by

$$\frac{dD_y^2}{dt} \propto \begin{cases} t^{(1-q)/(1+q)}\displaystyle\int_0^{\infty} e^{-\zeta/(1+q)}S(\zeta)\,d\zeta & \text{for jet-type flows,} \\ t^{1-2q}\displaystyle\int_0^{\infty} e^{-(1-q)\zeta}S(\zeta)\,d\zeta & \text{for wake-type flows,} \end{cases}$$

where $S(\zeta)$ is the covariance function of $g(\eta)$. Thus, in all cases,

$$D_y(t) \propto D_x(t). \tag{7.1}$$

The proportionality of the longitudinal and lateral dispersions is a simple consequence of the fact that the transformations needed to convert the longitudinal and lateral components of the particle velocity into stationary random functions are of the same form. However it should be kept in mind that, as a result of the shearing action associated with the distribution of mean Eulerian velocity, the fluctuations in $u(t)$ are greater in magnitude, and possibly persist for a longer time, than those in $v(t)$, so that $D_x(t)$ may be a good deal larger numerically than $D_y(t)$. The increase in thickness of the shear layer with distance downstream is essentially a process of diffusive spreading of fluid marked with finite vorticity and, since the two lateral components of velocity have comparable intensities in two-dimensional shear layers, $D_y(t)$ is likely to be of the same general magnitude as the thickness of the shear layer at $x = \bar{X}(t)$.

The lateral spread of positions of the marked particle would be of particular interest in a case in which marked particles are released at the origin at regular intervals, one after the other, or when the release is continuous, as it might be when dissolved salt is used as the method of marking the fluid. The boundary, in the (x, y)-plane, of the 'wake' of marked fluid, defined as the curve on which the mean density of marked fluid is some low arbitrarily chosen fraction of the mean density at $y = 0$ for the same value of x, is then a curve whose ordinate is proportional to $D_y(t)$ when the abscissa has the value $\bar{X}(t)$. Thus the marked fluid is bounded by the curve

$$y \propto x = x^p \quad \text{for jet-type flows,}$$

or

$$y \propto x^{1-q} = x^p \quad \text{for wake-type flows.}$$

The thickness of the 'wake' of marked fluid in the y-direction is proportional, in all cases, to the thickness of the turbulent shear layer in the z-direction. The marked fluid spreads out in the z-direction as rapidly as the growth in thickness of the shear layer allows it, so that when marked fluid is released continuously the 'wake' of marked fluid has the same cross-sectional shape at all values of x.

8. Some common features of the above results

Despite the apparent differences between the various cases of free turbulent shear flow, some aspects of the results obtained in §§ 5–7 are common to them all. Two of these common features in particular call for notice in view of their fundamental character. The first is the relation between t and the new variable η. For both jet-type and wake-type flows, this relation was found to be

$$\eta = \log t; \tag{8.1}$$

this was also the relation found in the case of decaying turbulence behind a grid (Batchelor & Townsend 1956). Going back one step, the common feature of all these cases of turbulent flow which leads to this logarithmic relation is the fact that the time scale characterizing the motion of a fluid particle is proportional to t, where t is measured from the instant at which the particle is released (at the point at which the turbulence originates).

It is remarkable that in such widely different types of turbulent flow as jets, mixed layers, wakes, and decaying grid turbulence, the time scale of oscillations in the velocity of a fluid particle should always increase linearly with t. The distinctive property common to all these developing or decaying turbulent flows is their self-preservation, or similarity of structure at different stations normal to the mean streamlines, and one is led to enquire if a simple dimensional argument will yield the result about the time scale. Presumably the explanation lies in the fact that the interval t since the marked particle was released at the origin is also the interval of time since the turbulent eddy which surrounds the particle was generated at the origin $x = 0$. Eddies are generated with infinite energy and zero linear dimensions at the (virtual) origin, and then convected downstream, and the development of the free turbulent shear layer as a function of x is essentially a process of diffusion and decay of the eddies under the action of inertia forces. In the absence of more than one dimensional parameter specifying the conditions of generation of the eddies at the origin (such as the momentum flux in the case of a jet), the time scale or representative period of the eddies moving downstream must, for dimensional reasons, be proportional to the decay time t. This representative period of the eddies is identical with the time scale of fluctuations of the velocity of an element of fluid in the eddy, and so the above general result is recovered.

The second interesting common feature of the results of § 5 and § 6 is that

$$D_x(t) \propto [\overline{X}(t)]^p; \tag{8.2}$$

for both jet-type and wake-type flows the longitudinal dispersion increases in proportion to the thickness of the shear layer at the mean position of the particle. (The same is true of $D_y(t)$ in cases of two-dimensional flow, as a consequence of the proportionality between $D_x(t)$ and $D_y(t)$.) This can be regarded as essentially a product of a dimensional argument, although such an argument might not be convincing by itself in view of doubt about whether the thickness of the shear layer at the mean position of the particle is really the only length available as a measure of the dispersion. (Results appropriate to a case in which the marked fluid particle is released at some finite value of x may also be obtained with analysis like that already described. The same forms of $D_x(t)$, etc., as those given above are found for sufficiently large values of t, but the results for smaller values of t depend on the position of release and are certainly not obtainable from dimensional arguments.)

9. The maximum mean concentration of marked fluid

As the marked particle moves downstream the statistical dispersion of its position increases, in the way already described, and the probability of finding the marked particle in unit volume located at its mean position decreases correspondingly. This probability of finding the marked particle in unit volume at any given position and time, which may be termed the mean concentration of marked fluid, has its absolute maximum value at some constant value of $(y/x^p, z/x^p)$ which is unknown (although for any flow which is symmetrical the maximum clearly lies on the centre line), and at a distance downstream $x = \overline{X}(t)$ which increases with t. It is implicit in the hypothesis made in § 4 that the whole probability distribution of the displacement $X(t)$ attains a self-preserving form, and the same will be true of the displacements in the two lateral directions; thus the maximum mean concentration of marked fluid, $C_m(t)$ say, may be determined from the relation expressing conservation of the total amount of marked fluid. The linear extent of the distribution of mean concentration of marked fluid in the x-direction is measured by $D_x(t)$, in any lateral direction in which the turbulence is of finite extent by $[\overline{X}(t)]^p$, and in any lateral direction in which the turbulence is stationary by $D_y(t)$. Hence

$$C_m(t) \,.\, D_x(t) \,.\, [\overline{X}(t)]^{2p} = \text{constant}$$

in cases in which the turbulence is of finite extent in the two lateral directions (as in a round jet), and

$$C_m(t) \,.\, D_x(t) \,.\, D_y(t) \,.\, [\overline{X}(t)]^p = \text{constant}$$

in cases of two-dimensional mean flow. In all cases we have

$$D_x(t) \propto D_y(t) \propto [\overline{X}(t)]^p,$$

so that the variation of $C_m(t)$ can be written generally as

$$C_m(t) \propto [\overline{X}(t)]^{-3p}. \tag{9.1}$$

If the release of marked particles at $x = 0$ is continuous, the distribution of mean concentration of marked fluid is steady and the maximum mean concentration (now only a maximum with respect to y and z) is a function of x alone, say $C'_m(x)$. For turbulence which is of finite extent in the two lateral directions, uniformity of the flux of marked fluid across different sections downstream requires

$$C'_m(x) \propto \begin{cases} x^{q-2p} = x^{q-2} & \text{for jet-type flows,} \\ x^{-2p} \;\;\, = x^{2q-2} & \text{for wake-type flows.} \end{cases} \tag{9.2}$$

For turbulence which is stationary in one lateral direction, the diffusion in this lateral unbounded direction must be taken into account, but, as already seen, the dispersion of a particle in this lateral direction is proportional to the thickness of the shear layer at the mean position of the particle, and the above functional forms of $C'_m(x)$ are again applicable. (These results for $C'_m(x)$ can also be obtained by integrating the contributions from a whole set of instantaneous sources of marked fluid, released at different times, provided the fact that the similarity laws are applicable to wake-type flows only when x and t are large is employed.)

10. Table of results for the various free shear flows

The values of the similarity powers p and q for the cases of the two-dimensional jet, wake and mixing layer (with zero velocity in the stream on one side) and the axi-symmetrical jet and wake are well known (Goldstein 1938). Substitution in the formulae (4.4), (5.1), (5.5), (6.4), (9.1) and (9.2) then gives the power laws shown in table 1.

	Round jet	Plane jet	Mixing layer	Round wake	Plane wake
Eulerian length scale $L(x)$	1	1	1	1	$\frac{1}{2}$
Eulerian velocity scale $V(x)$	-1	$-\frac{1}{2}$	0	$-\frac{2}{3}$	$-\frac{1}{2}$
Mean displacement of particle $\overline{X}(t)$	$\frac{1}{2}$	$\frac{2}{3}$	1	1	1
Lagrangian velocity scale $\omega(t)$	$-\frac{1}{2}$	$-\frac{1}{3}$	0	$-\frac{2}{3}$	$-\frac{1}{2}$
Lagrangian time scale $\tau(t)$	1	1	1	1	1
Dispersion $D_x(t)$, $D_y(t)$	$\frac{1}{2}$	$\frac{2}{3}$	1	$\frac{1}{3}$	$\frac{1}{2}$
Maximum mean concentration (release of one particle) $C_m(t)$	$-\frac{3}{2}$	-2	-3	-1	$-\frac{3}{2}$
Maximum mean concentration (continuous release) $C'_m(x)$	-1	$-\frac{3}{2}$	-2	$-\frac{2}{3}$	-1

Table 1. Showing powers of the independent variable (either x or t) for the quantities listed on the left hand side.

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